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VIBRATION OF CIRCULAR PLATES ON A FREE FLUID SURFACE: EFFECT OF SURFACE WAVES

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The effect of free-surface waves on free vibrations of circular plates resting on a free fluid surface is studied. The solution is achieved by using a perturbation technique and the Hankel transformation method which give a couple of dual integral equations of Titchmarsh type. The fluid is considered inviscid and incompressible and the velocity potential describes its motion. The Kirchhoff theory of plates is used to model the elastic thin plate. The theory is suitable for all axisymmetric plate boundary conditions. Numerical results are given in non-dimensional form for modes up to five nodal circles and diameters for clamped, simply supported and free-edge plates, to be ready-to-use in applications. The effect of free-surface waves on the plate's natural frequencies is significant only when bluging and sloshing modes of the system have close natural frequencies.

1. INTRODUCTION

Several studies have been addressed in the past to vibrations of circular plates in stationary heavy fluids (liquids). After the poineering work of Lord Rayleigh [1], the first study on this topic can be attributed to Lamb [2]. He studied the free vibrations of clamped, circular baffled plates using simple assumed modes and an approximation to obtain the hydrodynamic pressure; this solution was extended to free-edge circular plates by McLachlan [3]. Amabili and Kwak [4] have recently solved the same problem using a refined approach. The effect of a finite fluid depth above the baffled plate was investigated by Amabili [5]. Amabili *et al.* [6] extended the study in reference [4] to annular, baffled plates. For this class of problems, the boundary conditions on the fluid domain are homogeneous and give a Neumann problem.

Elastic circular bottom plates in fluid-filled cylindrical tanks have been largely studied in relation to free vibrations and sloshing in the container; e.g., see references [7–17]. In this case, the fluid velocity potential can be obtained using the method of separation of variables.

Free vibrations of circular plates resting on a free fluid surface have been studied for the first time a few years ago by Kwak and Kim [18] for axisymmetric modes and by Kwak [19] for the general case. These studies also address the circular plates completely submerged in an infinite fluid domain. Experiments confirming the results of references [18, 19] have been performed by Amabili *et al.* [20]. Kwak and Amabili [21] extended this study to annular plates, successfully comparing theoretical and experimental results. In all these studies, the boundary conditions on the fluid domain are mixed and give a Dirichlet problem. In particular, a zero-velocity potential is imposed at the free liquid surface, so that the effect of free-surface waves is neglected.

Other interesting studies on vibrations of circular plates in stationary fluids can be found in the literature; see, e.g., references [22–27].

The present study is addressed to free vibrations of circular plates resting on a free fluid surface, which is the same problem already solved by Kwak [19]; however in this study, the effect of free-surface waves is retained. This effect was previously investigated but only for bottom plates of cylindrical tanks, e.g. see references [7, 13, 16]. In particular, in a fluid–structure system having a free surface and under gravity, two families of modes appear: sloshing and bulging modes. Sloshing modes are due to oscillation of the fluid and usually have low frequency. Bulging modes are due to the structural vibration; they are affected by the free surface oscillation and the presence of fluid–structure interface. Here, attention is focused on bulging modes. The solution is achieved using a perturbation technique and the Hankel transformation method which give a couple of dual integral equations of Titchmarsh type. The fluid is considered inviscid and incompressible and the velocity potential describes its motion. The Kirchhoff theory of plates is used to model the elastic thin plate. The theory is suitable for all axisymmetric plate boundary conditions. Numerical results are given in the non-dimensional form for modes up to five nodal circles and diameters for clamped, simply supported and free-edge plates, to be ready-to-use in applications.

2. MIXED BOUNDARY VALUE PROBLEM

A polar co-ordinate system (O, r, θ) is introduced with the origin at O the centre of the plate. The mode shapes, related to transverse deflection w, for *in vacuo* free vibrations of thin elastic circular plates are expressed by

$$w(r, \theta, t) = W_{nm}(r)\cos(n\theta)\sin(\omega t), \tag{1}$$

where $W_{nm}(r) = [\mathrm{J}_n(\lambda_{nm}r/a) + \alpha_{nm}\mathrm{I}_n(\lambda_{nm}r/a)]$, n is the number of nodal diameters, m is the number of nodal circles, ω is the radian frequency and λ_{nm} and α_{nm} are the frequency and mode-shape parameters, respectively, both dependent on the plate's boundary conditions. J_n and I_n are the Bessel function and the modified Bessel function of order n, respectively. The equations that give λ_{nm} and α_{nm} are reported in section 6; some of these data can be found in reference [28]. The radian *in vacuo* frequency of vibration ω_V and the frequency parameter λ_{nm} are related by $\omega_V = \lambda_{nm}^2\sqrt{D/(\rho_P h a^4)}$, where $D = (Eh^3)/[12(1-\nu^2)]$, ρ_P is the plate mass density, h is the plate thickness, a is the plate radius, E is Young's modulus and ν is the Poisson ratio. In equation (1) the Kirchhoff theory of plates [29] was used.

The hypothesis is made that the mode-shapes of vibrating plate on the fluid surface are the same as *in vacuo*. This means that the dynamic loading of the fluid is assumed to have a negligible effect on the natural mode-shapes of the plate. Experimental tests, performed by Montero de Espinosa and Gallego-Juàrez [22], Amabili *et al.* [20] and Amabili and Kwak [21] show that mode shapes are little modified by the presence of water, especially for free-edge plates. Amabili and Kwak [4] theoretically computed the actual mode shapes of a baffled circular plate in contact with liquid on one side. They showed that the actual mode shapes are quite close to the ones *in vacuo*, and for free-edge plates the differences are particularly small. With this assumption, equation (1) is also valid for a plate on the fluid surface.

If the fluid is incompressible and inviscid and its movement irrotational, it is possible to describe the fluid motion (due to the plate's vibration) by the velocity potential Φ that must satisfy the Laplace equation $\nabla^2\Phi = 0$. The fluid velocity is given by $\mathbf{v} = -\nabla\Phi$. By using the variable separation with respect to the angular co-ordinate, Φ can be expressed as

$$\Phi(r, \theta, z, t) = \phi(r, z)\cos(n\theta)\,\omega\cos(\omega t), \tag{2}$$

where ϕ satisfies

$$\frac{\partial^2\phi}{\partial r^2} + \frac{\partial\phi}{r\partial r} + \frac{\partial^2\phi}{\partial z^2} - \frac{n^2}{r^2}\phi = 0 \quad \text{in the fluid domain} \tag{3}$$

and ω is the radian frequency of the fluid–plate system. Then, the following is imposed: (i) a contact without cavitation at the fluid–plate interface S_B; (ii) the linearized free-surface condition on the free fluid surface S_F and (iii) the radiation condition at an infinite distance from the plate S_R (see Figure 1). The superficial

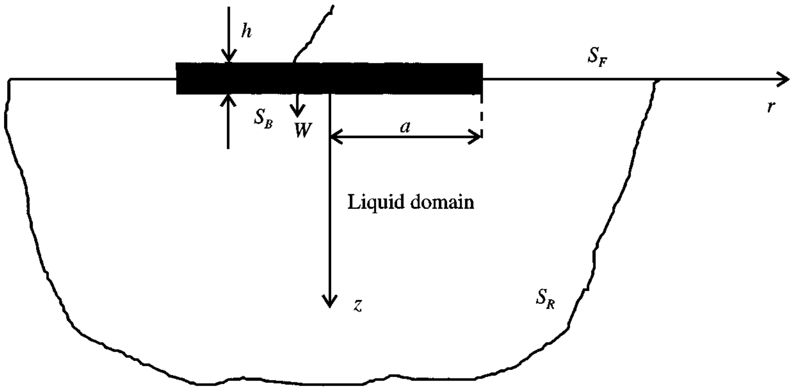

Figure 1. Circular plate on a free fluid surface; co-ordinate system and symbols.

tension of the fluid is neglected. Therefore, the boundary conditions can be expressed as

$$\frac{\partial \phi}{\partial z} = -W_{nm}(r) \quad \text{on } S_B, \qquad \phi - \frac{g}{\omega^2}\frac{\partial \phi}{\partial z} = 0 \quad \text{on } S_F, \qquad \phi, \frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial z} \to 0 \quad \text{on } S_R, \tag{4–6}$$

where g is the gravity acceleration. To solve the mixed boundary value problem it is useful to use the modified Hankel transformation, as addressed by Kwak and Kim [18] and Kwak [19]. It is defined as

$$\bar{\phi}_h = \int_0^\infty r\phi(r, z)\mathrm{J}_n(\xi r)\,\mathrm{d}r. \tag{7}$$

Applying the Hankel transform to the first, second and fourth term of equation (3), one can find

$$\int_0^\infty r\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{n^2}{r^2}\phi\right)\mathrm{J}_n(\xi r)\,\mathrm{d}r = -\xi^2\int_0^\infty r\phi(r, z)\mathrm{J}_n(\xi r)\,\mathrm{d}r = -\xi^2\bar{\phi}_h(\xi, z). \tag{8}$$

Using equation (3), the result of equation (8) must be equal to

$$-\int_0^\infty r\frac{\partial^2 \phi}{\partial z^2}\mathrm{J}_n(\xi r)\,\mathrm{d}r = -\frac{\mathrm{d}^2}{\mathrm{d}z^2}\bar{\phi}_h(\xi, z). \tag{9}$$

Therefore equation (3) is reduced to the following ordinary differential equation:

$$\mathrm{d}^2\bar{\phi}_h/\mathrm{d}z^2 - \xi^2\bar{\phi}_h = 0. \tag{10}$$

The general solution of equation (10) that satisfies equation (6) is

$$\bar{\phi}_h(\xi, z) = B(a\xi)\mathrm{e}^{-\xi z}. \tag{11}$$

The investigation formula for the Hankel transform gives

$$\phi(r, z) = \int_0^\infty \xi\bar{\phi}_h(\xi, z)\mathrm{J}_n(\xi r)\,\mathrm{d}\xi. \tag{12}$$

3. PERTURBATION APPROACH AND SUPERPOSITION PRINCIPLE

To solve the problem, it is useful to introduce the perturbation parameter

$$\varepsilon = g/a\omega^2 \tag{13}$$

which makes it possible to write the function ϕ as the sum of two contributions

$$\phi = \phi_0 + \varepsilon\phi_1, \tag{14}$$

where ϕ_0 is the contribution related to the following mixed boundary value problem in which free-surface waves are neglected [19],

$$\partial\phi_0/\partial z = -W_{nm}(r) \quad \text{on } S_B, \qquad \phi_0 = 0 \quad \text{on } S_F \tag{15}$$

and ϕ_1 is the contribution related to the following mixed boundary value problem in which the plate is considered rigid and only sloshing of the free surface is considered:

$$\partial\phi_1/\partial z = 0 \quad \text{on } S_B, \qquad \phi_1 = a\,\partial\phi_0/\partial z \quad \text{on } S_F \tag{16}$$

The second of equations (16) was obtained by neglecting the terms in ε^2 in the free-surface condition. By using the principle of superposition, the sum of the two mixed boundary value problems gives the actual one. This method is described in detail in reference [30]. The perturbation parameter ε is introduced in the second term of equation (14) to indicate that it gives a small contribution with respect to the first one when we are studying the bulging modes (where the plate oscillates, moving the liquid).

Upon substituting equations (11) and (12) in equations (15) and (16), two sets of dual integral equations of Titchmarsh type [31] are obtained,

$$\int_0^\infty \xi^2 B_0(a\xi)\mathrm{J}_n(\xi r)\,\mathrm{d}\xi = W_{nm}(r), \quad 0 < r \leqslant a,$$

$$\int_0^\infty \xi B_0(a\xi)\mathrm{J}_n(\xi r)\,\mathrm{d}\xi = 0, \quad r > a, \tag{17}$$

$$\int_0^\infty \xi^2 B_1(a\xi)\mathrm{J}_n(\xi r)\,\mathrm{d}\xi = 0, \quad 0 < r \leqslant a,$$

$$\int_0^\infty \xi B_1(a\xi)\mathrm{J}_n(\xi r)\,\mathrm{d}\xi = -\,a\int_0^\infty \xi^2 B_0(a\xi)\mathrm{J}_n(\xi r)\,\mathrm{d}\xi, \quad r > a, \tag{18}$$

where the unknown functions B_0 and B_1 are related to the two mixed boundary value problems given by equations (15) and (16) respectively.

Upon introducing the non-dimensional variables

$$\rho = r/a, \qquad \eta = a\xi, \qquad A_0(\eta) = \eta B_0(\eta), \qquad A_1(\eta) = \eta B_1(\eta), \tag{19}$$

the transformed equations become

$$\int_0^\infty \eta A_0(\eta)\mathrm{J}_n(\rho\eta)\,\mathrm{d}\eta = a^3 W_{nm}(\rho), \quad 0 < \rho \leqslant 1,$$

$$\int_0^\infty A_0(\eta)\mathrm{J}_n(\rho\eta)\,\mathrm{d}\eta = 0, \quad \rho > 1, \tag{20}$$

$$\int_0^\infty \eta A_1(\eta)\mathrm{J}_n(\rho\eta)\,\mathrm{d}\eta = 0, \quad 0 < \rho \leqslant 1,$$

$$\int_0^\infty A_1(\eta)\mathrm{J}_n(\rho\eta)\,\mathrm{d}\eta = -\int_0^\infty \eta A_0(\eta)\mathrm{J}_n(\rho\eta)\,\mathrm{d}\eta, \quad \rho > 1. \tag{21}$$

4. KINETIC ENERGIES OF THE FLUID AND THE PLATE

In section 2, it is assumed that the wet mode shapes are the same as the mode shapes *in vacuo*, so that there is no change in kinetic and elastic potential energies of the plate. For a plate vibrating *in vacuo*, one can write

$$\omega_V^2 = V_P/T_P^*, \tag{22}$$

where V_P is the maximum potential energy of the plate and T_P^* is its reference kinetic energy. For a plate vibrating on the fluid free surface one has [30, 32]

$$\omega_F^2 = (V_P + V_F)/(T_P^* + \tilde{T}_F^*), \tag{23}$$

where V_P and T_P^* are the same as in equation (22), V_F is the maximum potential energy associated to free surface waves and $\tilde{T}_F^*$ is the reference kinetic energy of the fluid. It is well-known that by using Green's theorem [33, 30] it is possible to evaluate the reference kinetic energy of the fluid with a surface integral on the boundary of the fluid domain, i.e. on $S_B + S_F$ in this case, as a consequence that the integral on S_R is zero. Amabili [30] proved that equation (23) can be simplified into

$$\omega_F^2 = V_P/(T_P^* + T_F^*), \tag{24}$$

where T_F^* is the reduced reference kinetic energy of the fluid computed by integrating only over the wet plate surface S_B.

By using the non-dimensional variables, the potential at the free surface can be expressed as

$$\phi(\rho, 0) = \frac{1}{a^2}\int_0^\infty A(\eta)\mathrm{J}_n(\rho\eta)\,\mathrm{d}\eta. \tag{25}$$

The reduced reference kinetic energy of the fluid is expressed as

$$\begin{aligned} T_F^* &= -\frac{1}{2}\rho_F a^2 D_n \int_0^1 \phi(\rho, 0)\frac{\partial\phi(\rho, 0)}{\partial z}\rho\mathrm{d}\rho \\ &= \frac{1}{2}\rho_F a^2 D_n\left[\int_0^1 \phi_0(\rho, 0)W_{nm}(\rho)\,\rho\mathrm{d}\rho + \varepsilon\int_0^1 \phi_1(\rho, 0)W_{nm}(\rho)\,\rho\mathrm{d}\rho\right] \\ &= T_{F0}^* + \varepsilon T_{F1}^*, \end{aligned} \tag{26}$$

where

$$T_{F0}^* = \frac{1}{2}\rho_F a^2 D_n \int_0^1 \phi_0(\rho, 0)W_{nm}(\rho)\,\rho\mathrm{d}\rho, \tag{27}$$

$$T_{F1}^* = \frac{1}{2}\rho_F a^2 D_n \int_0^1 \phi_1(\rho, 0)W_{nm}(\rho)\,\rho\mathrm{d}\rho, \tag{28}$$

and

$$D_n = \begin{Bmatrix} 2\pi & \text{if } n = 0, \\ \pi & \text{if } n \geqslant 1. \end{Bmatrix}$$

T^*_{F0} and T^*_{F1} represent the zero order and the first order kinetic energies of the fluid, respectively.

To simplify the formulation one can introduce the new variables

$$\delta(\rho) = \int_0^\infty \eta A(\eta) \mathrm{J}_n(\rho\eta)\,\mathrm{d}\eta, \qquad \chi(\rho) = \int_0^\infty A(\eta)\mathrm{J}_n(\rho\eta)\,\mathrm{d}\eta. \tag{29, 30}$$

Then, the potential at the free surface can be written as

$$\phi(\rho, 0) = (1/a^2)\chi(\rho), \tag{31}$$

and the dual integral equations can be rewritten as

$$\begin{aligned} &\delta_{0B}(\rho) = a^3 W_{nm}(\rho), \quad 0 < \rho \leqslant 1, \qquad &&\delta_{1B}(\rho) = 0, \qquad &&0 < \rho \leqslant 1, \\ &\chi_{0F}(\rho) = 0, \qquad \rho > 1, &&\chi_{1F}(\rho) = -\delta_{0F}(\rho), &&\rho > 1. \end{aligned} \tag{32, 33}$$

The first subscript to δ and χ refers to the variables B_0 and B_1 and the second to the domains S_B and S_F. To calculate the kinetic energy T^*_F, the terms $\chi_{0B}(\rho)$, $\chi_{1B}(\rho)$, $\delta_{0F}(\rho)$ need to be determined.

Based on the result of Sneddon [31] applied to equation (32), one can write

$$\chi_{0B}(\rho) = \frac{2}{\pi}\rho^n \int_\rho^1 \frac{u^{-2n} F^*(u)\,\mathrm{d}u}{\sqrt{u^2-\rho^2}}, \tag{34}$$

where

$$F^*(u) = a^3 \int_0^u \frac{v^{n+1} W_{nm}(v)\,\mathrm{d}v}{\sqrt{u^2 - v^2}} = a^3 u^{n+1} \int_0^1 \frac{y^{n+1} W_{nm}(uy)\,\mathrm{d}y}{\sqrt{1-y^2}}. \tag{35}$$

Using the above result, one can derive the zero order kinetic energy of the fluid

$$\begin{aligned} T^*_{F0} &= \frac{1}{2}\rho_F D_n \int_0^1 \chi_{0B}(\rho) W_{nm}(\rho)\,\rho\,\mathrm{d}\rho \\ &= \frac{1}{2}\rho_F D_n a^3 \frac{2}{\pi}\int_0^1 u^2 \left(\int_0^1 \frac{y^{n+1} W_{nm}(uy)\,\mathrm{d}y}{\sqrt{1-y^2}}\right)^2 \mathrm{d}u. \end{aligned} \tag{36}$$

The above calculation was carried out by Kwak [19]. The final result has the form [34]

$$T^*_{F0} = \frac{\rho_F D_n a^3}{2\lambda_{nm}} \int_0^1 u[\mathrm{J}_{n+1/2}(\lambda_{nm}u) + \alpha_{nm}\mathrm{I}_{n+1/2}(\lambda_{nm}u)]^2\,\mathrm{d}u = \frac{1}{2}\rho_F D_n a^3 \Delta_{F0}, \tag{37}$$

where

$$\Delta_{F0} = (1/2\lambda_{nm})[\delta_{F1} + 2\alpha_{nm}\delta_{F2} + \alpha_{nm}^2 \delta_{F3}], \tag{38}$$

in which

$$\delta_{F1} = \mathrm{J}^2_{n+1/2}(\lambda_{nm}) - \mathrm{J}_{n-1/2}(\lambda_{nm})\mathrm{J}_{n+3/2}(\lambda_{nm}),$$

$$\delta_{F2} = \frac{1}{\lambda_{nm}}[\mathrm{J}_{n+1/2}(\lambda_{nm})\mathrm{I}_{n-1/2}(\lambda_{nm}) - \mathrm{J}_{n-1/2}(\lambda_{nm})\mathrm{I}_{n+1/2}(\lambda_{nm})],$$

$$\delta_{F3} = \mathrm{I}^2_{n+1/2}(\lambda_{nm}) - \mathrm{I}_{n-1/2}(\lambda_{nm})\mathrm{I}_{n+3/2}(\lambda_{nm}).$$

To calculate the first order kinetic energy of the water, one needs to calculate δ_{0F} and χ_{1B}. Following the result of Sneddon [31], one obtains

$$\delta_{0F} = -a^3 \frac{2\rho^{-n}\sqrt{\rho^2-1}}{\pi} \int_0^1 \frac{u^{n+1}\sqrt{1-u^2}W_{nm}(u)\,\mathrm{d}u}{\rho^2-u^2} \tag{39}$$

and

$$\begin{aligned}\chi_{1B}(\rho) &= -\frac{2\sqrt{1-\rho^2}\,\rho^n}{\pi} \int_1^\infty \frac{t^{1-n}\delta_{0F}(t)\,\mathrm{d}t}{(t^2-\rho^2)\sqrt{t^2-1}} \\ &= a^3 \frac{4\sqrt{1-\rho^2}\,\rho^n}{\pi^2} \int_1^\infty \frac{t^{1-2n}}{t^2-\rho^2} \int_0^1 \frac{u^{n+1}\sqrt{1-u^2}W_{nm}(u)\,\mathrm{d}u\,\mathrm{d}t}{t^2-u^2}.\end{aligned} \tag{40}$$

Inserting equation (40) into equation (28), one obtains

$$\begin{aligned}T^*_{F1} &= \frac{1}{2}\rho_F D_n \int_0^1 \chi_{1B}(\rho) W_{nm}(\rho)\,\rho\mathrm{d}\rho \\ &= \frac{1}{2}\rho_F D_n a^3 \frac{4}{\pi^2} \int_1^\infty t^{1-2n} \left[\int_0^1 \frac{u^{n+1}\sqrt{1-u^2}W_{nm}(u)\,\mathrm{d}u}{t^2-u^2}\right]^2 \mathrm{d}t.\end{aligned} \tag{41}$$

Upon introducing a dummy variable, $t = 1/\rho$ to reduce the integal on infinite domain into one on finite domain, the first order kinetic energy can be expressed as

$$T^*_{F1} = \frac{1}{2}\rho_F D_n a^3 \Delta_{F1}, \tag{42}$$

where

$$\Delta_{F1} = \frac{4}{\pi^2} \int_0^1 \rho^{2n+1} U^2_{nm}(\rho)\,\mathrm{d}\rho \tag{43}$$

in which

$$U_{nm}(\rho) = \int_0^1 \frac{u^{n+1}\sqrt{1-u^2}\,W_{nm}(u)\,\mathrm{d}u}{1-\rho^2u^2}.$$

It is impossible to express $T^*_{F_1}$ in closed form, because the integral in equation (43) must be performed numerically.

The kinetic energy of the plate can be expressed as

$$T^*_P = \frac{1}{2}\rho_P h D_n a^2 \Delta_P, \tag{44}$$

where

$$\Delta_P = \frac{1}{2}[\delta_{P1} + 2\alpha_{nm}\delta_{P2} + \alpha^2_{nm}\delta_{P3}] \tag{45}$$

in which

$$\delta_{P1} = J_n^2(\lambda_{nm}) - J_{n-1}(\lambda_{nm})J_{n+1}(\lambda_{nm}),$$

$$\delta_{P2} = \frac{1}{\lambda_{nm}}[J_n(\lambda_{nm})I_{n-1}(\lambda_{nm}) - J_{n-1}(\lambda_{nm})I_n(\lambda_{nm})],$$

$$\delta_{P3} = I_n^2(\lambda_{nm}) - I_{n-1}(\lambda_{nm})I_{n+1}(\lambda_{nm}).$$

5. NON-DIMENSIONAL PARAMETERS

Using equations (22), (24) and (26), one can derive

$$\omega_V^2 = \left(1 + \frac{T_{F0}^*}{T_P^*} + \varepsilon\frac{T_{F1}^*}{T_P^*}\right)\omega_F^2, \tag{46}$$

where ω_V is the radian frequency of the plate *in vacuo* and ω_F is the radian frequency of the plate in contact with the fluid surface. It is useful to introduce the non-dimensional parameters

$$\frac{T_{F0}^*}{T_P^*} = \left(\frac{\rho_F a}{\rho_P h}\right)\frac{\Delta_{F0}}{\Delta_P} = \beta\Gamma_{nm} \tag{47}$$

in which $\beta = \rho_F a/\rho_P h$ is the so-called thickness correction factor and $\Gamma_{nm} = \Delta_{F0}/\Delta_P$ is the nondimensionalized added virtual mass incremental (NAVMI) factor [18, 19] or zero order NAVMI factor. In addition, one can derive

$$\varepsilon\frac{T_{F1}^*}{T_P^*}\omega^2 = \frac{g}{a\omega_V^2}\beta\frac{\Delta_{F1}}{\Delta_P}\omega_V^2 = \varepsilon_V\beta\Pi_{nm}\omega_V^2, \tag{48}$$

where $\varepsilon_V = g/a\omega_V^2$ is defined as the non-dimensionalized free-surface effect factor and $\Pi_{nm} = \Delta_{F1}/\Delta_P$ is defined as the first order non-dimensionalized added virtual mass incremental factor. Hence, one can express the frequency change in the form

$$\frac{\omega_F}{\omega_V} = \sqrt{\frac{1 - \varepsilon_V\beta\Pi_{nm}}{1 + \beta\Gamma_{nm}}}. \tag{49}$$

6. NUMERICAL RESULTS AND DISCUSSION

Initially, the frequency and mode-shape parameters for plates *in vacuo* have been calculated. The frequency parameters of circular plates having clamped, simply supported, and free-edge boundary conditions can be obtained by solving the characteristic equations

$$J_n(\lambda_{nm})I_{n+1}(\lambda_{nm}) + I_n(\lambda_{nm})J_{n+1}(\lambda_{nm}) = 0 \quad \text{for a clamped circular plate,}$$

$$\frac{J_{n+1}(\lambda_{nm})}{J_n(\lambda_{nm})} + \frac{I_{n+1}(\lambda_{nm})}{J_n(\lambda_{nm})} = \frac{2\lambda_{nm}}{1-\nu} \quad \text{for a simply supported circular plate}$$

and

$$\frac{\lambda_{nm}^2 \mathrm{J}_n(\lambda_{nm}) + (1-\nu)[\lambda_{nm}\mathrm{J}_n'(\lambda_{nm}) - n^2\mathrm{J}_n(\lambda_{nm})]}{\lambda_{nm}^2 \mathrm{I}_n(\lambda_{nm}) - (1-\nu)[\lambda_{nm}\mathrm{I}_n'(\lambda_{nm}) - n^2\mathrm{I}_n(\lambda_{nm})]}$$

$$= \frac{\lambda_{nm}^2 \mathrm{J}_n'(\lambda_{nm}) + (1-\nu)n^2[\lambda_{nm}\mathrm{J}_n'(\lambda_{nm}) - \mathrm{J}_n(\lambda_{nm})]}{\lambda_{nm}^2 \mathrm{I}_n'(\lambda_{nm}) - (1-\nu)n^2[\lambda_{mn}\mathrm{I}_n'(\lambda_{nm}) - \mathrm{I}_n(\lambda_{nm})]} \quad \text{for a free-edge circular plate,}$$

where J_n' and I_n' indicate the derivatives of Bessel functions with respect to the argument.

The mode-shape parameter can be calculated by using the equations

$$\alpha_{nm} = -\,\mathrm{J}_n(\lambda_{nm})/\mathrm{I}_n(\lambda_{nm}) \quad \text{for a clamped and simply supported circular plate}$$

and

$$\alpha_{nm} = \frac{\lambda_{nm}^2 \mathrm{J}_n(\lambda_{nm}) + (1-\nu)[\lambda_{nm}\mathrm{J}_n'(\lambda_{nm}) - n^2\mathrm{J}_n(\lambda_{nm})]}{\lambda_{nm}^2 \mathrm{I}_n(\lambda_{nm}) - (1-\nu)[\lambda_{mn}\mathrm{I}_n'(\lambda_{nm}) - n^2\mathrm{I}_n(\lambda_{nm})]} \quad \text{for a free-edge circular plate.}$$

The frequency parameters λ_{nm} and the zero-order NAVMI factors Γ_{nm} are given in Appendix A for n and m up to five and for free-edge, simply supported and clamped plates. The first order NAVMI factors are given in Tables 1, 2 and 3 for free-edge, simply supported and clamped plates respectively. These coefficients are

TABLE 1

First order NAVMI factors Π_{nm} for clamped circular plates

m	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0	0·05336	0·01078	0·004294	0·002254	0·001377	0·0009227
1	0·01721	0·005973	0·002925	0·001746	0·001142	0·0008011
2	0·008202	0·003696	0·002099	0·001344	0·0009295	0·0006784
3	0·004767	0·002502	0·001555	0·001057	0·0007628	0·0005748
4	0·003109	0·001804	0·001194	0·0008501	0·0006344	0·0004904
5	0·002186	0·001363	0·0009467	0·0006974	0·0005346	0·0004224

TABLE 2

First order NAVMI factors Π_{nm} for simply supported circular plates ($\nu = 0{\cdot}3$)

m	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0	0·07055	0·01706	0·007603	0·004314	0·002788	0·001953
1	0·01263	0·005882	0·003442	0·002274	0·001620	0·001216
2	0·005298	0·003059	0·002024	0·001448	0·001092	0·0008550
3	0·002908	0·001886	0·001344	0·001013	0·0007942	0·0006409
4	0·001836	0·001282	0·0009608	0·0007517	0·0006064	0·0005008
5	0·001264	0·0009289	0·0007223	0·0005812	0·0004795	0·0004032

TABLE 3

First order NAVMI factors Π_{nm} for free-edge circular plates ($\nu = 0{\cdot}3$)

m	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	—	—	0·01922	0·01326	0·009991	0·007938
1	0·004209	0·001298	0·0008576	0·0007460	0·0006920	0·0006500
2	0·0009123	0·0004691	0·0003261	0·0002707	0·0002452	0·0002310
3	0·0004064	0·0002463	0·0001797	0·0001478	0·0001310	0·0001213
4	0·0002311	0·0001533	0·0001161	0·0000960	0·0000843	0·0000770
5	0·0001493	0·0001051	0·0000820	0·0000684	0·0000600	0·0000543

ready-to-use in applications, in conjunction with equation (49), due to their non-dimensional form.

Figures 2–4 show the ratio ω_F/ω_V versus the non-dimensionalized free-surface effect factor ε_V for the first modes with $m = 0$ of free-edge, simply supported and clamped plates, respectively. These figures are otained from equation (49) for $\beta = 50$. They show that the effect of the liquid on natural frequencies decrease with n and that, for $\beta = 50$, it is very large for all the modes considered. In particular, it is larger for simply supported and clamped plates with respect to free-edge plates. Moreover, this effect increases with ε_V; it is to note that $\varepsilon_V = 0$ corresponds to the case studied by Kwak [19] where free-surface waves are neglected. For the modes considered, a value of ε_V in the range between 0·1 and 0·5 corresponds to a significant effect of free-surface waves on the natural frequencies of the system. For $\varepsilon_V < 0{\cdot}1$ and $\beta = 50$, the effect of free-surface waves is negligible for most of the modes; this means that, for sufficiently high natural frequencies of the plate *in vacuo* and a radius not too small, the effect of free-surface waves is negligible.

The effect of the thickness correlation factor β on the ratio ω_F/ω_V is shown in Figure 5, where curves for three different values of β are given for the first axisymmetric mode ($n = 0$, $m = 0$) of simply supported circular plates. Figure 5 shows that the effect of free surface waves is increased significantly for large values of β.

To understand better the effect of free-surface waves, it is useful to compare the present result ω_F with the radian frequency obtained neglecting the effect of free surface waves, $(\omega_F)_{NW}$, i.e. computed for $\varepsilon_V = 0$ and already studied by Kwak [19]. It is also useful to introduce the sloshing frequency of the fluid (liquid) in order to compare it with the natural frequency of the bulging modes (the only ones considered in the present study) of the system. In particular, the sloshing radian frequency of an undisturbed free liquid surface of infinite depth and extension is given by

$$\omega_S^2 = gk_h, \tag{50}$$

where $k_h = \sqrt{k^2 + l^2}$ is the horizontal wave number and k and l are the (integer) wave numbers in two orthogonal directions on the free surface. Actually, the sloshing frequencies of the present system are different from the values computed

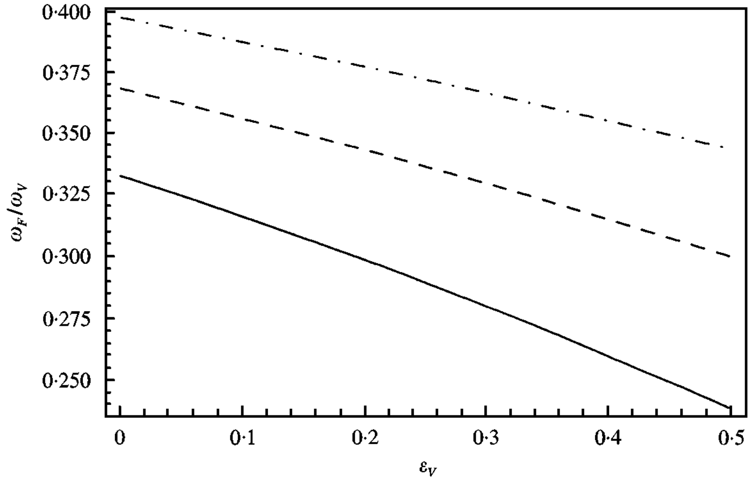

Figure 2. Ratio ω_F/ω_V versus ε_V for free-edge circular plates and $\beta = 50$. —— $n = 2$, $m = 0$; – – – $n = 3$, $m = 0$; – · – · – $n = 4$, $m = 0$.

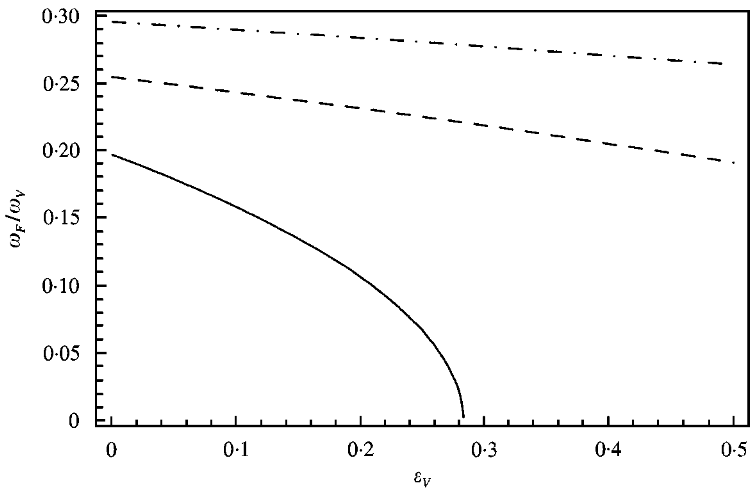

Figure 3. Ratio ω_F/ω_V versus ε_V for simply supported circular plates and $\beta = 50$. —— $n = 0$, $m = 0$; – – – $n = 1$, $m = 0$; – · – · – $n = 2$, $m = 0$.

by equation (50) as a consequence of the presence of the plate on the surface and the circular shape of waves instead of the linear one assumed to obtain expression (50). In any case, the lower frequency computed from equation (50), i.e. $\omega_S = \sqrt{g}$, can be used as representative of the lower frequency range of sloshing modes.

The ratio $\omega_F/(\omega_F)_{NW}$ versus the ratio $\omega_S/(\omega_F)_{NW}$ is given in Figures 6–8, with the following material and geometric properties assumed: $E = 206 \times 10^9$ MPa, $\rho_P = 7850\ \mathrm{kg/m^3}$, $\rho_F = 1000\ \mathrm{kg/m^3}$, $v = 0{\cdot}3$ and $h = 0{\cdot}5$ mm. In particular, the ratio $\omega_F/(\omega_F)_{NW}$ indicates the effect of free surface waves on natural frequencies of the system and the ratio $\omega_S/(\omega_F)_{NW}$ shows the difference between the natural

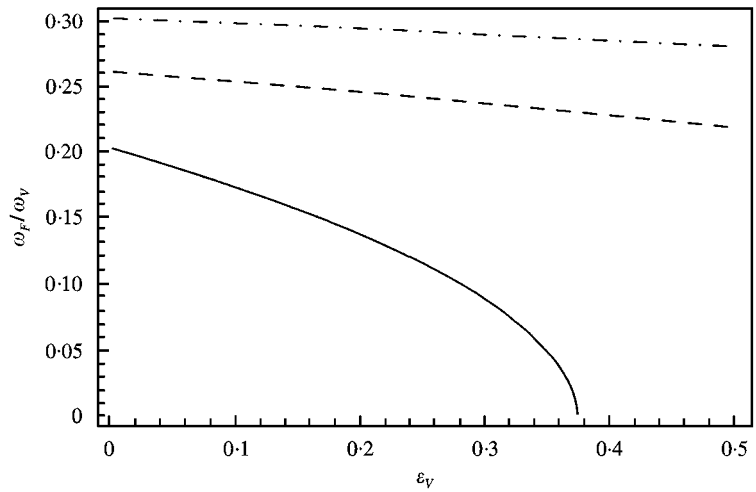

Figure 4. Ratio ω_F/ω_V versus ε_V for clamped circular plates and $\beta = 50$. —— $n = 0$, $m = 0$; – – – $n = 1$, $m = 0$; – · – · – $n = 2$, $m = 0$.

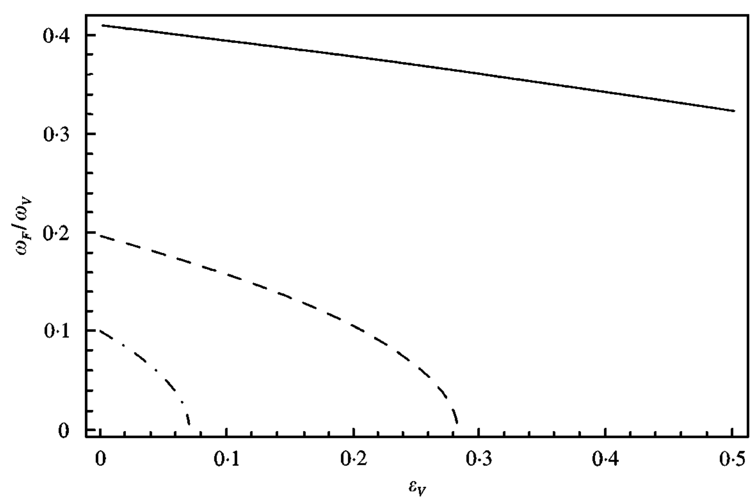

Figure 5. Ratio ω_F/ω_V versus ε_V for the first axisymmetric mode ($n = 0$, $m = 0$) of simply supported plates. —— $\beta = 10$; – – – $\beta = 50$; – · – · – $\beta = 200$.

frequency of the wet plate (neglecting waves) $(\omega_F)_{NW}$ for the mode considered and the sloshing frequency ω_S of the fluid. Figures 6–8 show that when the natural frequency of the wet plate (neglecting waves) $(\omega_F)_{NW}$ approaches the sloshing frequency ω_S, the effect of free surfaces waves is significant. In contrast, when $(\omega_F)_{NW}$ is much larger than ω_S the effect of free-surface waves on the natural frequency of the system is negligible. In particular, the figures show that the first axisymmetric mode ($n = 0$, $m = 0$) of the simply supported circular plate is the one more affected by free surface waves. Moreover, for $\omega_S/(\omega_F)_{NW} = 1$ the effect of free surface waves is varying from significant to very large for all the modes considered.

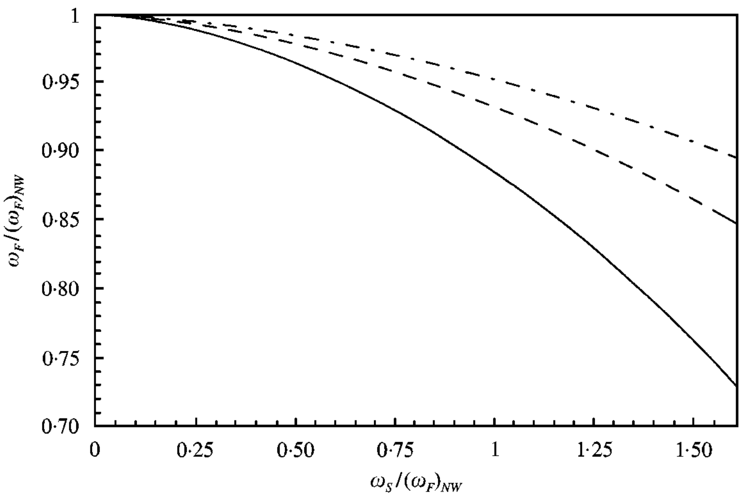

Figure 6. Ratio $\omega_F/(\omega_F)_{NW}$ versus $\omega_S/(\omega_F)_{NW}$ for free-edge circular plates. —— $n = 2$, $m = 0$; – – – $n = 3$, $m = 0$; – · – · – $n = 4$, $m = 0$.

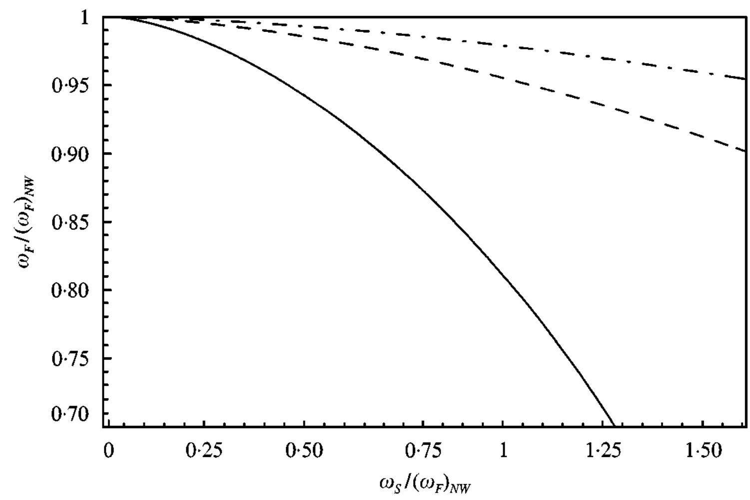

Figure 7. Ratio $\omega_F/(\omega_F)_{NW}$ versus $\omega_S/(\omega_F)_{NW}$ for simply supported circular plates. —— $n = 0$, $m = 0$; – – – $n = 1$, $m = 0$; – · – · – $n = 2$, $m = 0$.

7. CONCLUSIONS

The results show that the effect of free-surface waves significantly affects the natural frequencies of the bulging modes (where the plate oscillates, thus moving the fluid) only for very flexible plates, i.e. plates having fundamental frequency quite close to frequencies of surface waves. However, when this is verified, the interaction between sloshing and bulging modes is very large and the natural frequencies of bulging modes are largely decreased. It is to note that the present perturbation

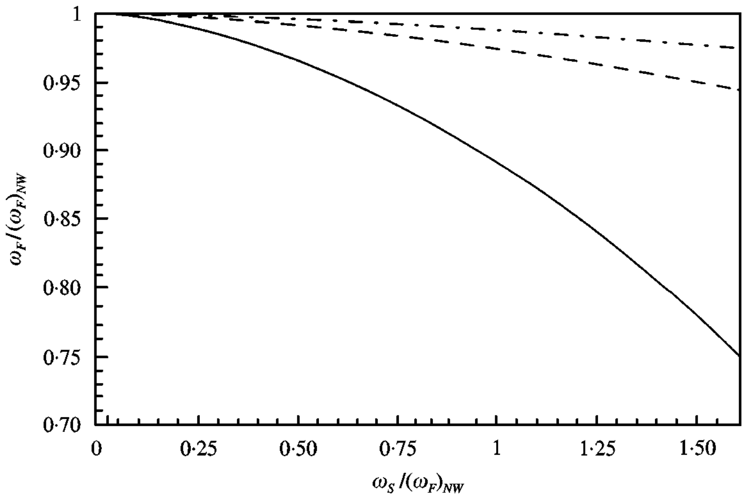

Figure 8. Ratio $\omega_F/(\omega_F)_{NW}$ versus $\omega_S/(\omega_F)_{NW}$ for clamped circular plates. —— $n = 0$, $m = 0$; – – – $n = 1$, $m = 0$; – · – · – $n = 2$, $m = 0$.

approach is based on the hypothesis that ε is small (i.e. that ε^2 is negligible with respect to ε). As a consequence of a direct relation between ε and ε_V, the present results can be considered accurate only when ε_V is sufficiently small.

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APPENDIX A: ZERO-ORDER NAVMI FACTORS AND FREQUENCY PARAMETERS

In this appendix, the zero order NAVMI factors Γ_{nm} and the frequency parameters λ_{nm} of circular plates are given in Tables 4–9 for three different

TABLE 4

Zero order NAVMI factors Γ_{nm} for clamped circular plates

m	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0	0·4667	0·2723	0·1986	0·1583	0·1324	0·1142
1	0·2032	0·1523	0·1239	0·1055	0·09233	0·08235
2	0·1265	0·1045	0·08996	0·07945	0·07142	0·06503
3	0·09121	0·07922	0·07047	0·06372	0·05832	0·05388
4	0·07118	0·06368	0·05786	0·05318	0·04931	0·04603
5	0·05830	0·05319	0·04906	0·04562	0·04271	0·04020

TABLE 5

Zero order NAVMI factors Γ_{nm} for simply supported circular plates ($\nu = 0{\cdot}3$)

m	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0	0·4946	0·2874	0·2083	0·1651	0·1375	0·1182
1	0·1958	0·1509	0·1243	0·1064	0·09332	0·08335
2	0·1217	0·1025	0·08910	0·07916	0·07143	0·06520
3	0·08818	0·07755	0·06953	0·06320	0·05805	0·05376
4	0·06911	0·06238	0·05703	0·05264	0·04896	0·04582
5	0·05680	0·05217	0·04835	0·04513	0·04236	0·03996

TABLE 6

Zero order NAVMI factors Γ_{nm} for free-edge circular plates ($\nu = 0{\cdot}3$)

m	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0	—	—	0·1612	0·1275	0·1065	0·09192
1	0·2181	0·1727	0·1386	0·1159	0·09986	0·08800
2	0·1371	0·1135	0·09717	0·08523	0·07610	0·06888
3	0·09630	0·08386	0·07451	0·06720	0·06131	0·05647
4	0·07403	0·06642	0·06037	0·05544	0·05133	0·04785
5	0·06009	0·05496	0·05074	0·04719	0·04415	0·04153

TABLE 7

Frequency parameters λ_{nm} for clamped circular plates

m	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	3·1962	4·6109	5·9059	7·1442	8·3466	9·5257
1	6·3064	7·7987	9·2114	10·5361	11·8367	13·1074
2	9·4395	10·9581	12·4020	13·7949	15·1499	16·4751
3	12·5771	14·1089	15·5792	17·0050	18·3960	19·7583
4	15·7164	17·2560	18·7451	20·1921	21·6084	22·9979
5	18·8565	20·4010	21·9009	23·3660	24·8015	26·2117

TABLE 8

Frequency parameters λ_{nm} for simply supported circular plates ($\nu = 0{\cdot}3$)

m	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	2·2215	3·7280	5·0610	6·3212	7·5393	8·7294
1	5·4516	6·9627	8·3736	9·7236	11·0319	12·3093
2	8·6114	10·1377	11·5887	12·9875	14·3475	15·6773
3	11·7609	13·2967	14·7717	16·2014	17·5957	18·9613
4	14·9069	16·4489	17·9399	19·3910	20·8098	22·2018
5	18·0513	19·5977	21·1001	22·5670	24·0042	25·4164

TABLE 9

Frequency parameters λ_{nm} for free-edge circular plates ($\nu = 0{\cdot}3$)

m	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	—	—	2·3148	3·5269	4·6728	5·7875
1	3·0005	4·5249	5·9380	7·2806	8·5757	9·8364
2	6·2003	7·7338	9·1851	10·5804	11·9344	13·2565
3	9·3675	10·9068	12·3817	13·8091	15·1997	16·5606
4	12·5227	14·0667	15·5575	17·0070	18·4232	19·8117
5	15·6727	17·2203	18·7226	20·1882	21·6234	23·0330

boundary conditions: (i) clamped plates in Tables 4 and 7 (ii) simply supported plates in Tables 5 and 8; (iii) free-edge plates in Tables 6 and 9. Coefficients given in Tables 4–9 have been recalculated for the present study and can be satisfactorily compared with those published by Kwak [19] and Amabili *et al.* [20].